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COHERENT-STATE PATH-INTEGRAL APPROACH FOR CONSTRAINED FERMION SYSTEMS

GEORG JUNKER

Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstr. 7, D-91058 Erlangen, Germany
E-mail: junker@theorie1.physik.uni-erlangen.de

JOHN R. KLAUDER

Departments of Physics and Mathematics, University of Florida, Gainesville, FL-32611, USA
E-mail: klauder@phys.ufl.edu

The coherent-state path-integral representation for the propagator of fermionic systems subjected to first-class constraints is constructed. As in the bosonic case the importance of path-integral measures for Lagrange multipliers is emphasized. One example is discussed in some detail.

1 Introduction

The quantization of constrained systems is of considerable importance in many areas of theoretical physics. At the last meeting of this conference series, held at Dubna (Russia) in 1996, one of us has reexamined this problem from the point of view of coherent-state path integrals.[1,2] It is the aim of this contribution to show that this work on bosonic systems can be generalized to systems with fermionic degrees of freedom. Here we will limit ourselves to a brief discussion of first-class constraints. More details, including also an extensive discussion of second-class constraints, have already been published elsewhere.[3] Similar work has also been studied by Prokhorov and Shabanov.[4]

The fermionic quantum systems considered here consist of N fermionic degrees of freedom characterized by annihilation and creation operators obeying the anticommutation relations $\{f_i, f_j\} = 0$, $\{f_i^\dagger, f_j^\dagger\} = 0$, $\{f_i^\dagger, f_j\} = \delta_{ij}$ and acting on the "N-fermion" Hilbert space $\mathcal{H} \equiv \mathbb{C}^{2^N} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. The quantum dynamics of such a system is completely characterized by a normal-ordered self-adjoint Hamiltonian $H = H(f^\dagger, f)$ and the associated constraints, which are denoted by $\Phi_a(f^\dagger, f)$ and $\chi_\alpha(f^\dagger, f)$ for constraints being even and odd in the fermion operators, respectively. Here f and $f^\dagger$ stand for the ordered set $\{f_1, \ldots, f_N\}$ and $\{f_1^\dagger, \ldots, f_N^\dagger\}$, whereas a and α enumerate the even and odd constraints, respectively. As for the Hamiltonian, we assume that the operator-valued constraints are self-adjoint and normal ordered. The associated normalized fermionic coherent states $|\Psi\rangle \equiv |\psi_1 \cdots \psi_N\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$ are labeled by odd Grassmann variables, satisfying $\psi_i^2 = 0 = \bar{\psi}_i^2$, and fulfill the relation $\langle \Psi''|O(f^\dagger, f)|\Psi'\rangle = O(\bar{\Psi}'', \Psi')\langle \Psi''|\Psi'\rangle$ for any normal-ordered operator O. We also note that these states form an over-complete set, $\langle \Psi''|\Psi'\rangle = \exp\{-\frac{1}{2}\bar{\Psi}''\cdot(\Psi''-\Psi') + \frac{1}{2}(\bar{\Psi}''-\bar{\Psi}')\cdot\Psi'\}$, where $\bar{\Psi}''\cdot\Psi' \equiv \bar{\psi}_1''\psi_1' + \cdots + \bar{\psi}_N''\psi_N'$ etc., and admit a resolution of unity in $\mathcal{H}$, $\int d\bar{\Psi}d\Psi\, |\Psi\rangle\langle\Psi| = 1$, based on the standard definition of Grassmann integration.[3,5]

2 First-class Constraints

In this contribution we limit ourselves to the case of first-class constraints, i.e. the constraints together with the Hamiltonian are assumed to close a Lie superalgebra (c, d, g, h and k denote complex-valued structure constants)

$$[\Phi_a, \Phi_b] = i\, c_{ab}{}^c \Phi_c\,, \quad [\Phi_a, \chi_\alpha] = i\, d_{a\alpha}{}^\beta \chi_\beta\,, \quad \{\chi_\alpha, \chi_\beta\} = i\, g_{\alpha\beta}{}^a \Phi_a\,, \quad (1)$$

$$[\Phi_a, H] = i\, h_a{}^b \Phi_b\,, \quad [\chi_\alpha, H] = i\, k_\alpha{}^\beta \chi_\beta\,. \quad (2)$$

The physical subspace of the Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{H}_{\rm phys} = \{|\varphi\rangle \in \mathcal{H} \,:\, \Phi_a|\varphi\rangle = 0 \text{ and } \chi_\alpha|\varphi\rangle = 0 \text{ for all } a \text{ and } \alpha\}\,. \quad (3)$$

However, from the last relation in (1) it is obvious that the even constraints, $\Phi_a|\varphi\rangle = 0$ for all a, imply the odd constraints $\chi_\alpha|\varphi\rangle = 0$ and, therefore, it suffices to consider only the Lie algebra, c.f. the first relation in (1), of the even constraints generating the Lie group G. As a consequence, this subset of (boselike) constraints may now be treated in analogy to the bosonic case.[1,2] That is, we introduce the projection operator (μ denotes the invariant normalized Haar measure on G)

$$\mathbb{E} \equiv \int_G d\mu(\xi)\, \exp\{-i\xi^a \Phi_a\} = \mathbb{E}^\dagger = \mathbb{E}^2 = \exp\{-i\eta^a \Phi_a\}\mathbb{E}\,, \quad (4)$$

where the last relation is valid for any values of the real group parameters η. Clearly, the above definition only holds for compact groups G. However, even in the case of non-compact groups we may strictly follow the methods designed in the bose case.[2] Obviously, we have $\mathcal{H}_{\rm phys} = \mathbb{E}\mathcal{H}\mathbb{E}$ and the constrained time-evolution operator admits the following representations (by using the first relation in (2) and the last one in (4):

$$\exp\{-itH\}\mathbb{E} = \mathbb{E}\exp\{-itH\} = \mathbb{E}\exp\{-itH\}\mathbb{E} = \mathbb{E}\exp\{-it(\mathbb{E}H\mathbb{E})\}\mathbb{E}\,. \quad (5)$$

For the fermion coherent-state matrix element of this operator, that is, the propagator $\langle\Psi''|\exp\{-itH\}\mathbb{E}|\Psi'\rangle$, we will now construct a path-integral representation again following closely the previous approach.[2]

3 Path-Integral Representation of the Constrained Propagator

In order to construct a path-integral representation for the constrained propagator we start with the last relation in (4), divide the time interval t into M short-time intervals $\varepsilon = t/M$ and then use the group composition law following from the first relation in (2):

$$e^{-itH}\mathbb{E} = e^{-itH}e^{-i\xi^a\Phi_a}\mathbb{E} = \prod_{n=1}^{M} e^{-i\varepsilon H}e^{-i\varepsilon\eta^a\Phi_a}\mathbb{E}\,. \quad (6)$$

A time-lattice path-integral representation of the corresponding propagator then immediately follows from an M-fold insertion of the resolution of unity with fermion coherent states and the limit $\varepsilon \to 0$ such that $M\varepsilon = t$ remains fixed ($\Psi_M \equiv \Psi''$):

$$\langle\Psi''|\exp\{-itH\}\mathbb{E}|\Psi'\rangle = \lim_{\varepsilon\to 0}\prod_{n=0}^{M-1}\int d\bar{\Psi}_n d\Psi_n \int_G d\mu(\xi)\,\langle\Psi_0|e^{-i\xi^a\Phi_a(f^\dagger,f)}|\Psi'\rangle$$
$$\times \prod_{n=1}^{M}\exp\{-\tfrac{1}{2}\bar{\Psi}_n\cdot(\Psi_n-\Psi_{n-1}) + \tfrac{1}{2}(\bar{\Psi}_n-\bar{\Psi}_{n-1})\cdot\Psi_{n-1}$$
$$-i\varepsilon H(\bar{\Psi}_n,\Psi_{n-1}) - i\varepsilon\eta_n^a\Phi_a(\bar{\Psi}_n,\Psi_{n-1})\}\,. \quad (7)$$

Here we note that despite the fact that the time-dependent Lagrange multipliers η^a explicitly appear on the right-hand side the final result clearly does not depend on them. Hence, we are free to average this expression by any, in general complex-valued, normalized measure $\prod_n \int dC(\eta_n) = 1$. Therefore, we may express the above path integral formally by

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}C(\eta)\exp\left\{i\int_0^t d\tau\left[\tfrac{i}{2}(\bar{\Psi}\cdot\dot{\Psi} - \dot{\bar{\Psi}}\cdot\Psi) - H(\bar{\Psi},\Psi) - \eta^a\Phi_a(\bar{\Psi},\Psi)\right]\right\} \quad (8)$$

with the only requirement that the measure $\mathcal{D}C(\eta)$ should at least introduce one projection operator in order to respect the constraints.

4 Example

As an elementary but instructive example we choose two fermionic degrees of freedom $N = 2$ subjected to the constraint $\Phi(f^\dagger, f) = f_1^\dagger f_2 + f_2^\dagger f_1$ and a vanishing Hamiltonian $H = 0$. In this case the projector (4) has the following representations

$$\mathbb{E} = \frac{1}{2\pi}\int_0^{2\pi} d\xi\, e^{-i\xi\Phi} = 1 - \Phi^2 = 1 - f_1^\dagger f_1 - f_2^\dagger f_2 + 2f_1^\dagger f_1 f_2^\dagger f_2 \quad (9)$$

and its coherent-state matrix element $\langle\Psi''|\mathbb{E}|\Psi'\rangle$ is represented by the following path-integral ($\Delta\Psi_n = \Psi_n - \Psi_{n-1}$, $\Delta\bar{\Psi}_n = \bar{\Psi}_n - \bar{\Psi}_{n-1}$)

$$\begin{aligned}&\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}C(\eta)\exp\left\{i\int_0^t d\tau\left[\tfrac{i}{2}(\bar{\Psi}\cdot\dot{\Psi}-\dot{\bar{\Psi}}\cdot\Psi)-\eta(\bar{\psi}_1\psi_2+\bar{\psi}_2\psi_1)\right]\right\}\\ &=\lim_{\varepsilon\to 0}\prod_{n=0}^{M}\int d\bar{\Psi}_n d\Psi_n\prod_{n=1}^{M}\int_{-\infty}^{\infty}d\eta_n\,\delta(\eta_n)\int_0^{2\pi}\frac{d\xi}{2\pi}\langle\Psi_0|e^{-i\xi\Phi(f^\dagger,f)}|\Psi'\rangle\\ &\quad\times\prod_{n=1}^{M}\exp\left\{-\tfrac{1}{2}\bar{\Psi}_n\cdot\Delta\Psi_n+\tfrac{1}{2}\Delta\bar{\Psi}_n\cdot\Psi_{n-1}-i\varepsilon\eta_n\Phi(\bar{\Psi}_n,\Psi_{n-1})\right\}\ .\end{aligned}\tag{10}$$

Our choice for the measure of the Lagrange multipliers is obvious and obeys the requirement to insert at least one projection operator (here at the initial stage). Integration over the η's is trivial and for that over the Ψ's we use the convolution formula

$$\begin{aligned}&\int d\bar{\Psi}_n d\Psi_n\, e^{-\frac{1}{2}\bar{\Psi}_n\cdot\Delta\Psi_n+\frac{1}{2}\Delta\bar{\Psi}_n\cdot\Psi_{n-1}}\, e^{-\frac{1}{2}\bar{\Psi}_{n+1}\cdot\Delta\Psi_{n+1}+\frac{1}{2}\Delta\bar{\Psi}_{n+1}\cdot\Psi_n}\\ &\quad=\exp\left\{-\tfrac{1}{2}\bar{\Psi}_{n+1}\cdot(\Psi_{n+1}-\Psi_{n-1})+\tfrac{1}{2}(\bar{\Psi}_{n+1}-\bar{\Psi}_{n-1})\cdot\Psi_{n-1}\right\}\ .\end{aligned}\tag{11}$$

Finally, we note that ($|\Psi_0\rangle\equiv|\psi_1\rangle\otimes|\psi_2\rangle$)

$$\begin{aligned}&\langle\Psi_0|e^{-i\xi(f_1^\dagger f_2+f_2^\dagger f_1)}|\Psi'\rangle=\langle\Psi_0|\Psi'\rangle\\ &\quad\times\left[1+(\cos\xi-1)(\bar{\Psi}_0\cdot\Psi'+2\bar{\psi}_1\bar{\psi}_2\psi_1'\psi_2')-i\sin\xi\,(\bar{\psi}_1\psi_2'+\bar{\psi}_2\psi_1')\right]\ ,\end{aligned}\tag{12}$$

which immediately gives rise to the explicit result

$$\begin{aligned}\langle\Psi''|\mathbb{E}|\Psi'\rangle&=e^{-\frac{1}{2}\bar{\Psi}''\cdot\Psi''}e^{-\frac{1}{2}\bar{\Psi}'\cdot\Psi'}\left[1-3\bar{\psi}_1''\bar{\psi}_2''\psi_1'\psi_2'\right]\\ &=\langle\Psi''|\Psi'\rangle\left[1-\bar{\Psi}''\cdot\Psi'-2\bar{\psi}_1''\bar{\psi}_2''\psi_1'\psi_2'\right]\ .\end{aligned}\tag{13}$$

Further examples of first-class as well as second-class constraints can be found elsewhere.[3]

References

1. J.R. Klauder, in V.S. Yarunin and M.A. Smondyrev eds., *Path Integrals: Dubna '96*, (Joint Institut for Nuclear Research, Dubna, 1996) p. 51-60.
2. J.R. Klauder, Ann. Phys. (NY) **254** (1997) 419-453.
3. G. Junker and J.R. Klauder, Eur. Phys. J. C **4** (1998) 173-183.
4. L.V. Prokhorov and S.V. Shabanov, *Hamiltonian Mechanics for Gauge Systems*, (St. Petersburg Univ. Press, St. Petersburg, 1997), in Russian.
5. J.R. Klauder and B.-S. Skagerstam, *Coherent States*, (World Scientific, Singapore, 1985).